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FIXED POINT SETS OF DEFORMATIONS OF PAIRS OF SPACES*

Helga SCHIRMER
Carleton University, Ottawa, Canada

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Let (X, A) be a pair of compact polyhedra which satisfies conditions similar to those needed in order to realize the Nielsen number of the identity map in the cases where $A = \emptyset$ or $A = X$. Deformations of (X, A) with minimal fixed point sets are constructed, and are used to construct deformations of (X, A) with prescribed fixed point sets. The size and location of these fixed point sets depends only on the Euler characteristics of the components of X and A. As an application it follows that if M is an even-dimensional compact connected triangulable manifold with boundary Bd M, then there is no difference between possible fixed point sets of deformations of M and of deformations of $(M, \text{Bd } M)$.

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fixed points of deformations of pairs of spaces
realization of the Nielsen number
minimum number of fixed points of deformations
prescribed fixed points sets of deformations
complete invariance property for pairs

1. Introduction

The relative Nielsen number $N(f; X, A)$ of a selfmap $f: (X, A) \to (X, A)$ of a pair of compact ANR's was introduced in [6]. It has the standard properties of the Nielsen number $N(f)$ of a map $f: X \to X$, but is an improved and often best possible lower bound for the minimum number MF$[f; X, A]$ of fixed points of all maps in the homotopy class of f. (A homotopy of a map of pairs of spaces $f: (X, A) \to (X, A)$ is always understood to be a map of the form $H: (X \times I, A \times I) \to (X, A)$, where $I = [0, 1]$ and $H(x, 0) = f(x)$.) It was shown in [6, Theorem 6.2] that there exists a map homotopic to f with $N(f; X, A)$ fixed points, and hence that $N(f; X, A) =$ MF$[f; X, A]$, if (X, A) is a pair of compact polyhedra and if X and A satisfy certain conditions which extend those needed in the cases $A = \emptyset$ or $A = X$.

The Nielsen number $N(f)$ of a map $f: X \to X$ is equal to the minimum number MF$[f]$ of fixed points of all maps homotopic to f under weaker assumptions on X

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if f is a deformation (i.e. homotopic to the identity) rather than a map in an arbitrary homotopy class. In particular it is known that the Nielsen number $N(\mathrm{id})$ of the identity map $\mathrm{id}: X \to X$ equals $\mathrm{MF}[\mathrm{id}]$ if X is a 2-dimensionally connected polyhedron (as defined in Section 2), but not if X is an arbitrary compact polyhedron (see e.g. [1, p. 142 ff.; 7; 8; 10]).

Here we prove a similar result for maps of pairs of spaces. The Minimum Theorem 4.1 shows that $N(\mathrm{id}; X, A) = \mathrm{MF}[\mathrm{id}; X, A]$ if (X, A) is a '2-dimensionally connected pair of compact polyhedra', a condition which is e.g. satisfied if $X - A$ and all components of A are 2-dimensionally connected (see the definitions in Sections 2 and 3). But the Minimum Theorem 4.1 contains a stronger result, as it gives precise information about the possible location of minimal fixed point sets of deformations in terms of the Euler characteristics of X and A. If $A = \emptyset$, then the Nielsen number $N(\mathrm{id}; X, A) = N(\mathrm{id})$ of the identity map of a 2-dimensionally connected compact polyhedron X is zero if the Euler characteristic $\chi(X) = 0$ and is one if $\chi(X) \neq 0$. A deformation $f: X \to X$ with a minimal fixed point set is thus either fixed point free or has a single fixed point, and this fixed point can be located anywhere in X (see e.g. [1, Theorem 1, p. 142; 4, Theorem 3.1, p. 224]). The location of a minimal fixed point set of a deformation of a 2-dimensionally connected pair of compact polyhedra (X, A), however, can no longer be arbitrarily chosen, as it depends on the Euler characteristics of X and of the components of A. Details are given in Theorem 4.1.

The construction of a deformation $f: (X, A) \to (X, A)$ of a 2-dimensionally connected pair of compact polyhedra with a minimal fixed point set, which is required for the proof of Theorem 4.1, proceeds as usual in two steps. In the first f is homotoped to a fix-finite map g (i.e. to a map with a finite fixed point set), and as in [6, Theorem 4.1] we take care that g already has a minimal fixed point set on A. This step is carried out in Section 2. In the second step, which is contained in Section 3, fixed points lying in $X - A$ are united. Two isolated fixed points in $X - A$ can be united by standard methods, but it is also possible to unite a fixed point in $X - A$ with a fixed point on the boundary of A. This was proved in [6, Lemma 6.1] for arbitrary maps $f: (X, A) \to (X, A)$ under more restrictive assumptions on (X, A) and is here, in Lemma 3.2, carried out for deformations of 2-dimensionally connected pairs of compact polyhedra.

Deformations with minimal fixed point sets can be used to obtain maps with prescribed fixed point sets. A 2-dimensionally connected compact polyhedron X has the complete invariance property (CIP), which means that every closed nonempty subset F of X can be realized as the fixed point set $\mathrm{Fix}\, f = \{x \in X \mid f(x) = x\}$ of a map $f: X \to X$, and the construction of f is easy once a small deformation of X with precisely one fixed point located in F is known. ([4, Theorem 3.1], see also [5] for a survey of the CIP.) But it is no longer true that any closed nonempty subset F of X can be the fixed point set of a map $f: (X, A) \to (X, A)$ if (X, A) is a 2-dimensionally connected pair of compact polyhedra and $A \neq \emptyset$. In Theorem 5.1, which is an easy consequence of the Minimum Theorem 4.1 and of some preliminary results developed in the proof of Theorem 4.1, we obtain necessary and sufficient conditions for a set F so that F can be the fixed point set of a deformation $f:(X, A)\to(X, A)$. Remark 5.2 shows that these conditions can in general not be relaxed even if f is permitted to be a map of pairs in an arbitrary homotopy class.

The final Section 6 illustrates the results of this paper with two examples. One of them shows that if M is an even-dimensional manifold with boundary Bd M, then there is no difference between fixed point sets of deformations of M and those of $(M, \text{Bd } M)$.

We assume that the reader is familiar with [6]. Further background on the Nielsen number $N(f)$ of a map $f:X\to X$ can be found in [1] and [3].

2. Fix-finite deformations with minimal fixed point sets on a subspace

Let $f:(X, A)\to(X, A)$ be a selfmap of a pair of compact polyhedra and let $N(\bar{f})$ denote the Nielsen number of the restriction $\bar{f}:A\to A$ of f to A. Conditions on A are given in [6, Theorem 4.1] which ensure that f is homotopic to a map $g:(X, A)\to(X, A)$ which is fix-finite and has $N(\bar{f})$ fixed points on A. As expected, these conditions can be relaxed if f is a deformation. We will do this in Theorem 2.2, and we will also show that Fix $\bar{g}$ can be prescribed and that g can be chosen as a proximity map. This sharpened version of [6, Theorem 4.1] for deformations will be the starting point for the proofs of the results in Sections 4 and 5.

We denote by σ an open simplex of a simplicial complex K, by $\bar{\sigma}$ the corresponding closed simplex, and by $\kappa(x)$ the carrier simplex of a point x in K. A map $f:X_0\to|K|$ from a subset $X_0\subset|K|$ into the polyhedron $|K|$ is called a *proximity map* if $|\bar{\kappa}(x)|\cap|\bar{\kappa}(f(x))|\neq\emptyset$ for all $x\in X_0$. If $|L|$ is a subpolyhedron of $|K|$, then it follows from [1, pp. 124–125] that the track $\alpha(x, f(x), t)$ (where $t\in I$) of a homotopy between the identity map and a proximity map can be chosen as a broken line segment such that

$$\begin{cases} \alpha(x), f(x, t)=x & \text{for all } 0\leq t\leq 1 \text{ if } f(x)=x,\\ \alpha(x, f(x), t)\neq x & \text{for all } 0<t\leq 1 \text{ if } f(x)\neq x,\\ \alpha(x, f(x), t)\in|L| & \text{for all } 0\leq t\leq 1 \text{ if } x, f(x)\in|L|.\end{cases} \tag{2.1}$$

Hence a proximity map $f:(|K|, |L|)\to(|K|, |L|)$ is a deformation of $(|K|, |L|)$ (i.e. it is homotopic to the identity map $\text{id}:(|K|, |L|)\to(|K|, |L|)$), and it follows from the construction of α in [1] that each $f_t:(|K|, |L|)\to(|K|, |L|)$ given by $f_t(x)=\alpha(x, f(x), t)$ is again a proximity map.

If $H:X\times I\to Y$ is a homotopy, we write h_t for the map defined by $h_t(x)=H(x, t)$. A homotopy $H:A\times I\to X$, where $A\subset X$, is called *special* [2, p. 751] if the maps $h_t:A\to X$ have the same fixed point set for all $t\in I$. The proof of Theorem 2.2 will use the following lemma concerning homotopies and special homotopies of proximity maps.

Lemma 2.1. *Let $|K|$ be a compact polyhedron, let $|K_1|$ and $|K_2|$ be subpolyhedra with $|K_2| \subset |K_1|$, and let $f_0: |K_1| \to |K|$ be a proximity map.*

(i) *If $G: |K_2| \times I \to |K|$ is a homotopy of $g_0 = f_0 \mid |K_2|$ such that each $g_t: |K_2| \to |K|$ is a proximity map, then G extends to a homotopy $H: |K_1| \times I \to |K|$ of f_0 such that each h_t is a proximity map.*

(ii) *If, in addition, G is a special homotopy, then G extends to a special homotopy $F: |K_1| \times I \to |K|$ of f_0 such that each f_t is a proximity map.*

Proof. (i) Let the symbol $\sigma < \tau$, for $\sigma, \tau \in K$, denote that σ is a face of τ, and let $r: |K_1| \times I \to (|K_1| \times 0) \cup (|K_2| \times I)$ be a retraction so that $r(x, t) = (y, s)$ implies $\kappa(y) < \kappa(x)$ for all $(x, t) \in |K_1| \times I$. (The standard construction of r has this property (see e.g. [9, Cor. 4, p. 117]), if $H = G \circ r$, then for all $(x, t) \in |K_1| \times I$ we have $h_t(x) = g_s(y)$ with $\kappa(y) < \kappa(x)$. As g_s is a proximity map, $|\bar{\kappa}(y)| \cap |\bar{\kappa}(g_s(y))| \neq \emptyset$, and hence $|\bar{\kappa}(x)| \cap |\bar{\kappa}(h_t(x))| \neq \emptyset$. So each h_t is a proximity map.

(ii) F can now be constructed from $H = G \circ r$ as in the proof of [2, Lemma 2.1]. $\square$

A polyhedron $X = |K|$ is called 2-*dimensionally connected* if each maximal simplex is of dimension ≥ 2 and if there exists, for each pair of maximal simplexes $\sigma', \sigma'' \in K$, a chain of maximal simplexes $\sigma_0 = \sigma', \sigma_1, \ldots, \sigma_r = \sigma''$ so that $|\bar{\sigma}_{i-1}| \cap |\bar{\sigma}_i|$ is of dimension ≥ 1 for $i = 1, 2, \ldots, r$. We further define that the polyhedron $X = |K|$ is a *Nielsen space for deformations* if it admits a proximity map which is fixed point free if $\chi(X) = 0$ and which has one fixed point in an arbitrary location if $\chi(X) \neq 0$. (Compare the definition of a Nielsen space in [6, Section 4]. Examples of Nielsen spaces for deformations are 2-dimensionally connected compact polyhedra [4, Theorem 3.1], the circle and the arc, but not all compact polyhedra [8]. We write Cl Y, Int Y and Bd Y for the closure, interior and boundary in X of a subspace $Y \subset X$.

Theorem 2.2. *Let $(X, A) = (|K|, |L|)$ be a pair of compact polyhedra such that each component of A is a Nielsen space for deformations, and assume that A_j, with $j = 1, 2, \ldots, k$ $(k \geq 0)$, are the components of A with $\chi(A_j) \neq 0$. Then there exists a deformation g of (X, A) so that*

(i) *$\bar{g}$ has $k = N(\overline{\mathrm{id}})$ fixed points $a_1, a_2, \ldots, a_k$, and each a_j is a prescribed point of A_j,*

(ii) *g is fix-finite,*

(iii) *g is a proximity map with respect to the triangulation (K, L) of (X, A),*

(iv) *each fixed point of g on $X - A$ lies in a maximal simplex.*

Proof. The proof consists of a strengthening of the proof of [6, Theorem 4.1] to ensure that (iii) is true. As in [6], we proceed in two steps.

Step 1. We show that $\mathrm{id}: (X, A) \to (X, A)$ is homotopic to a map $h: (X, A) \to (X, A)$ which has the following properties:

(a) $\bar{h}$ has $N(\overline{\mathrm{id}})$ fixed points $a_1, a_2, \ldots, a_k$ which are prescribed as in (i);

(b) h is a proximity map with respect to the triangulation (K, L) of (X, A);

(c) there exists a compact polyhedron B in X so that $A \subset X - B$ and h is fixed point free on $\mathrm{Cl}(X - B) - A$.

As each component of A is a Nielsen space for deformations, we can homotope $\overline{\mathrm{id}}: A \to A$ to a map $\bar{h}: A \to A$ which is a proximity map with respect to the triangulation L of A and has $N(\overline{\mathrm{id}})$ fixed points $a_1, a_2, \ldots, a_k$ which are prescribed as in (i). If $\bar{H}: A \times I \to A$ is defined by $\bar{H}(x, t) = \alpha(x, \bar{h}(x), t)$ for all $(x, t) \in A \times I$, where α is as in (2.1), then $\bar{H}$ is a homotopy from $\overline{\mathrm{id}}$ to $\bar{h}$ and each $\bar{h}_t$ is a proximity map. As A is a strong deformation retract of some neighbourhood V of A in X [9, Cor. 11, p. 124] we can use a star covering of $A = |L|$ with respect to a subdivision of K to find a compact polyhedron A_1 with $A \subset \mathrm{Int}\, A_1 \subset A_1 \subset V$ so that $B = X - \mathrm{Int}\, A_1$ is a compact polyhedron in X. Let $R: A_1 \times I \to V$ be the restriction to A_1 of the deformation retraction of V onto A and let $r: A_1 \to A$ be the retraction given by $r(x) = R(x, 1)$. Then $\bar{H}: A \times I \to A$ defines $H_1: A_1 \times I \to X$ by

$$H_1(x, t) = \begin{cases} R(x, 2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \bar{H}(r(x), 2t-1) & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

and by choosing A_1 sufficiently close to A we can ensure that each restriction $H_1 | A_1 \times \{t\}$ is a proximity map with respect to (K, L). According to Lemma 2.1(i) we can extend H_1 to a homotopy $H: (X \times I, A \times I) \to (X, A)$ of the identity map such that each h_t is a proximity map. Hence the map $h = h_1: (X, A) \to (X, A)$ has the properties (a), (b) and (c).

Step 2. The map $h: (X, A) \to (X, A)$ is homotopic to a map $g: (X, A) \to (X, A)$ which has the properties (i)–(iv).

The proof of Step 2 is very similar to the proof of Step 2 of [6, Theorem 4.1]. If the homotopy G_B in [6] is chosen so that each restriction $G_B | B \times \{t\}$ is a proximity map with respect to (K, L) and if G_U is obtained with the help of Lemma 2.2 (ii), then the map g constructed in [6] has the properties (i)–(iv).

3. Uniting fixed points in the complement of a subspace

Our next goal is the construction of a deformation of a pair of compact polyhedra (X, A) which has a minimal fixed point set not only on A, but on all of X. This will be done by uniting the finitely many fixed points on $X - A$ of the map g constructed in Theorem 2.2 as far as possible. There are two cases: we can unite two fixed points which both lie in $X - A$, or we can unite a fixed point in $X - A$ with a fixed point on the boundary of A. The first case is known but will be proved in Lemma 3.1 in a stronger form, for we shall deal in the next two sections with prescribed as well as with minimal fixed point sets. The second case will be considered in Lemma 3.2.

Further assumptions on (X, A) are now needed. We define that $(X, A) = (|K|, |L|)$ is a 2-*dimensionally connected pair* of polyhedra if it is a pair of polyhedra so that

X is connected, $X-A$ is 2-dimensionally connected and each component of A is a Nielsen space for deformations. (If $A \neq \emptyset$ then $X-A$ is not a subpolyhedron of X, but the definition of 2-dimensional connectedness still applies.)

Lemma 3.1. *Let $(X, A)=(|K|,|L|)$ be a 2-dimensionally connected pair of compact polyhedra and let A_j, where $j=1,2,\ldots,k$ (with $k \geqslant 0$), be the components of A with $\chi(A_j) \neq 0$. Given a set $F=\{a_1, a_2, \ldots, a_k, x_0\}$, where each a_j is a prescribed point of A_j and x_0 is a prescribed point of $X-A$, there exists a proximity map $f:(X, A) \to (X, A)$ with either* $\operatorname{Fix} f=F$ *or* $\operatorname{Fix} f=F-\{x_0\}$.

Proof. According to Theorem 2.2 there exists a fix-finite map $g:(X, A) \to (X, A)$ with $\operatorname{Fix} g=\{a_1, a_2, \ldots, a_k\}$ which is a proximity map with respect to (K, L) and has all fixed points on $X-A$ located in maximal simplexes. If $\operatorname{Fix} g \cap (X-A)=\emptyset$, we are done. Otherwise we select a point $x' \in \operatorname{Fix} g \cap (X-A)$ and use [1, Lemma 3, p. 128; and, Lemma 2, p. 126] to unite the fixed points of g on $X-A$ with x' to construct a proximity map $g':(X, A) \to (X, A)$ with $\operatorname{Fix} g'=\{a_1, a_2, \ldots, a_k, x'\}$. Finally we use the method of the proof of [4, Lemma 2.4] in order to move x' to x_0, and thus obtain a proximity map $f:(X, A) \to (X, A)$ with $\operatorname{Fix} f=\{a_1, a_2, \ldots, a_k, x_0\}$. □

The next lemma extends the uniting of fixed points on $X-A$ and $\operatorname{Bd} A$, which is used in the proof of [6, Lemma 6.1] to a wider class of polyhedra in the case where f is a proximity map.

Lemma 3.2. *Let $(X, A)=(|K|,|L|)$ be a pair of compact polyhedra such that $X-A$ is 2-dimensionally connected, and let x_0 and x_1 be isolated fixed points of the proximity map $f:(X, A) \to (X, A)$. If x_0 lies in a maximal simplex of $X-A$ and x_1 on* $\operatorname{Bd} A$, *then f is homotopic to a proximity map $f':(X, A) \to (X, A)$ with* $\operatorname{Fix} f'=\operatorname{Fix} f-\{x_0\}$.

Proof. As in [1, Lemma 3, p. 128] we can move x_0 to any maximal simplex of $X-A$, and hence we can assume that x_0 lies in a maximal simplex $|\sigma|$ of $X-A$ with $x_1 \in |\bar{\sigma}|$. Let V be an open neighbourhood of the half-open segment $[x_0, x_1)$ so that $\operatorname{Cl} V$ is convex in $|\bar{\sigma}|$ with $\operatorname{Cl} V-\{x_1\} \subset |\sigma|$ and $\operatorname{Bd} V \cap A=\{x_1\}$ (see Fig. 1.) As $\operatorname{Cl} V \subset |\operatorname{st} \kappa(x_1)|$, there exists a $\delta>0$ with $\alpha(x, f(x), t) \in |\operatorname{st} \kappa(x_1)|$ for all $x \in \operatorname{Cl} V$ and $0 \leqslant t \leqslant \delta$. If we define a proximity map $f_1:(X, A) \to (X, A)$ by $f_1(x)=\alpha(x, f(x), \delta)$ for all $x \in X$, then (2.1) shows that $\operatorname{Fix} f_1=\operatorname{Fix} f$.

Now we label the points of $\operatorname{Cl} V-\{x_1\} \subset |\sigma|$ as $x=(b_x, t)$, where $b_x \in \operatorname{Bd} V-\{x_1\}$, $0<t_x \leqslant 1$ and $x=t_x b_x+(1-t_x) x_1$, and define $f':(X, A) \to (X, A)$ by

$$f'(x)=\begin{cases} x_1 & \text{if } x=x_1, \\ (1-t_x) x_1+t_x f_1(b_x) & \text{if } x=(b_x, t_x) \in \operatorname{Cl} V-\{x_1\}, \\ f_1(x) & \text{if } x \in X-V. \end{cases}$$

f' is well-defined as $b_x \in \operatorname{Bd} V-\{x_1\}$ implies $f_1(b_x) \in |\operatorname{st} \kappa(x_1)|$, and it is clearly continuous and a proximity map. As $f_1(b_x) \neq b_x$ for $b_x \in \operatorname{Bd} V-\{x_1\}$, the only fixed point of f' on $\operatorname{Cl} V$ is $\{x_1\}$. Hence f' satisfies Lemma 3.2. □

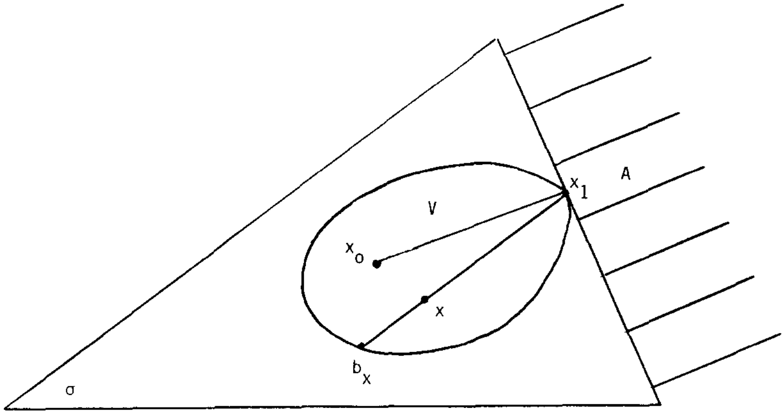

Fig. 1

4. Deformations of pairs of spaces with prescribed minimal fixed point sets

The main purpose of this section is to show that the relative Nielsen number $N(\mathrm{id}; X, A)$ of the identity map of a pair of spaces (X, A) is a sharp lower bound for the minimum number of fixed points in the homotopy class of $\mathrm{id}: (X, A) \to (X, A)$ under assumptions on (X, A) which are weaker than those made in [6, Theorem 6.1]. But we shall prove a stronger result: not only shall we construct a map which realizes $N(\mathrm{id}; X, A)$, but we shall also obtain precise information about the possible location of a minimal fixed point set.

The relative Nielsen number $N(\mathrm{id}; X, A)$ of the identity map $\mathrm{id}: (X, A) \to (X, A)$ of a pair of compact ANR's was calculated in [6, Theorem 2.6] and we have

$$N(\mathrm{id}; X, A) = \begin{cases} N(\overline{\mathrm{id}}) & \text{if } N(\overline{\mathrm{id}}) \neq 0, \\ N(\mathrm{id}) & \text{if } N(\overline{\mathrm{id}}) = 0, \end{cases} \tag{4.1}$$

where $N(\mathrm{id})$ is the Nielsen number of the identity map $\mathrm{id}: X \to X$ and $N(\overline{\mathrm{id}})$ the Nielsen number of its restriction $\overline{\mathrm{id}}: A \to A$ to A. The Nielsen numbers $N(\mathrm{id})$ and $N(\overline{\mathrm{id}})$ are known, as the Nielsen number of the identity map of a compact connected ANR X is 0 if $\chi(X) = 0$ and 1 if $\chi(X) \neq 0$ (see [1, Exercise 4, p. 96] or [3, Example 1, p. 18]).

Theorem 4.1 (Minimum Theorem). *If $(X, A) = (|K|, |L|)$ is a 2-dimensionally connected pair of compact polyhedra, then there exists a deformation f of (X, A) which has $N(\mathrm{id}; X, A)$ fixed points. They can be located as follows:*

(i) *If $\chi(X) = 0$ and $\chi(A_j) = 0$ for each component A_j of A, then f is fixed point free.*

(ii) *If $\chi(X) \neq 0$ but $\chi(A_j) = 0$ for each component A_j of A, then f has exactly one fixed point x_0. The point x_0 can be located anywhere in $X - \mathrm{Int}\, A$.*

(iii) *If $\chi(A_j) \neq 0$ for $k \geq 1$ components $A_1, A_2, \ldots, A_k$ of A, then f has exactly k fixed points $a_1, a_2, \ldots, a_k$. Each A_j, for $j = 1, 2, \ldots, k$, must contain one fixed point*

a_j. If $\chi(X)=\chi(A)$, then each a_j can be located anywhere in A_j, but if $\chi(X)\neq\chi(A)$, then at least one a_j must lie on Bd A.

In each case the deformation f can be chosen as a proximity map.

Proof. (i) According to Lemma 3.1 there exists a proximity map $g:(X, A)\to(X, A)$ with Fix $g=\emptyset$ or Fix $g=\{x_0\}$, where x_0 lies in a maximal simplex of $X-A$. In the first case we are done. In the second $\chi(X)=0$ implies that the Nielsen number $N(g)$ of the deformation $g:X\to X$ is zero. Hence the fixed point x_0 of g must be inessential, and so we can use [1, Theorem 4, p. 123] to homotope $g:(X, A)\to(X, A)$ to a fixed point free proximity map $f:(X, A)\to(X, A)$.

(ii) If $x_0\in X-A$, the existence of a proximity map $f:(X, A)\to(X, A)$ with fixed point set $\{x_0\}$ follows from Lemma 3.1 and the fact that $N(\mathrm{id})=1$ and hence Fix $f\neq\emptyset$. Now let x_0 be an arbitrary point of Bd A. In this case we choose a point y_0 in a maximal simplex of $X-A$, and use Lemma 3.1 and the fact that $N(\mathrm{id})=1$ to obtain a map $g:(X, A)\to(X, A)$ with Fix $g=\{y_0\}$. If we choose a bounded metric $\bar{d}\leq 1$ for X and define a proximity map $g':(X, A)\to(X, A)$ by

$$g'(x)=\alpha(x, g(x), \bar{d}(x, x_0)),$$

where α is as in (2.1), then Fix $g'=\{x_0, y_0\}$. Hence we can use Lemma 3.2 to homotope g' to the desired map f.

It remains to show that there exists no deformation $f:(X, A)\to(X, A)$ with Fix $f=\{x_0\}$ if $x_0\in$ Int A. We proceed indirectly, and assume that f is such a map. Then the fixed point index $\mathrm{ind}(f, x_0)=\mathrm{ind}(f, \mathrm{Int}\, A)$ of x_0 with respect to the map $f:X\to X$ is the same as the fixed point index $\mathrm{ind}(\bar{f}, x_0)=\mathrm{ind}(\bar{f}, \mathrm{Int}\, A)$ of x_0 with respect to the restriction $\bar{f}:A\to A$ of f. (See e.g. [1, p. 52 ff.] and [3, p. 11 ff.]) for the definition of the fixed point index.) But $\chi(X)\neq 0$ implies $N(f)=1$ and thus $\mathrm{ind}(f, x_0)\neq 0$, and $\chi(A_j)=0$ for all components A_j of A implies $N(\bar{f})=0$ and thus $\mathrm{ind}(\bar{f}, x_0)=0$. Therefore such a map f cannot exist.

(iii) In order to construct a desired map f, we choose arbitrary points $a_j\in A_j$ for $j=1, 2, \ldots, k$, but make sure that at least one a_j lies in Bd A if $\chi(X)\neq\chi(A)$. Then it follows from Lemma 3.1 that there exists a proximity map $g:(X, A)\to(X, A)$ with Fix $g=\{a_1, a_2, \ldots, a_k\}$ or Fix $g=\{a_1, a_2, \ldots, a_k, x_0\}$, where x_0 lies in a maximal simplex of $X-A$. In the first case we put $f=g$. In the second case, and if $a_{j_0}\in$ Bd A_{j_0} for at least one index j_0, we can use Lemma 3.2 to unite x_0 with $a_{j_0}\in$ Bd A_{j_0} in order to obtain the desired map f. If, finally, all $a_j\in$ Int A_j and hence $\chi(X)=\chi(A)$, then

$$\chi(A)=\sum_{j=1}^{k}\mathrm{ind}(\bar{g}, a_j)=\sum_{j=1}^{k}\mathrm{ind}(g, a_j)$$

and

$$\chi(X)=\sum_{j=1}^{k}\mathrm{ind}(g, a_j)+\mathrm{ind}(g, x_0),$$

therefore $\mathrm{ind}(g, x_0)=0$. So we can again obtain f from g with the help of [1, Theorem 4, p. 123].

We still have to show that the condition that $\{a_1, a_2, \ldots, a_k\} \cap \operatorname{Bd} A \neq \emptyset$ if $\chi(X) \neq \chi(A)$ is necessary. This follows again from an indirect argument. For if $f:(X, A) \to (X, A)$ is a deformation of (X, A) with $\operatorname{Fix} f = \{a_1, a_2, \ldots, a_k\}$, where $a_j \in \operatorname{Int} A_j$ for all $j = 1, 2, \ldots, k$, then $\operatorname{ind}(g, a_j) = \operatorname{ind}(\bar{g}, a_j)$ implies

$$\chi(X) = \sum_{j=1}^{k} \operatorname{ind}(g, a_j) = \sum_{j=1}^{k} \operatorname{ind}(\bar{g}, a_j) = \chi(A),$$

therefore such a map cannot exist if $\chi(X) \neq \chi(A)$. □

Remark 4.2. The assumptions of Theorem 4.1 are considerably weaker than those of the Minimum Theorem 6.2 in [6], where the relative Nielsen number $N(f; X, A)$ of a map in an arbitrary homotopy class was realized. In particular it is no longer necessary to assume that X is not a surface, nor that A can be by-passed in X. (See [6, Definition 5.1] for the definition of by-passing.) The fact that 2-dimensional connectedness of a compact polyhedral pair (X, A) does not imply that A can be by-passed in X can be seen from the 2-dimensionally connected pair (X, A) in Fig. 2, in which X is a disk with a hole H and A a triangle with the hole H located so that $\operatorname{Bd} X \cap \operatorname{Bd} A$ is a vertex of A. A cannot be by-passed in X, as the path p from x_0 to x_1 indicated in the figure is not homotopic with end points fixed to a path in $X - A$. But the assumptions of Theorem 4.1 cannot be relaxed much further, as can be seen by putting $A = \emptyset$ or $A = X$ and using [8].

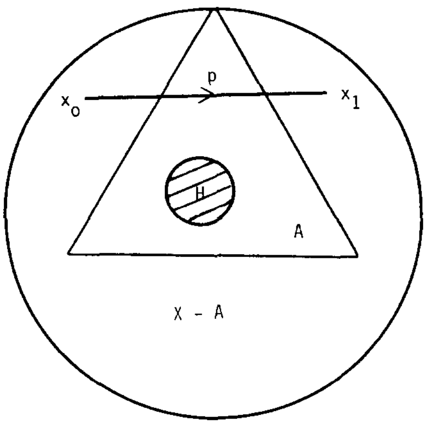

Fig. 2

5. Deformations of pairs of spaces with prescribed fixed point sets

The results of the previous two sections can be used to construct deformations of a 2-dimensionally connected pair of compact polyhedra (X, A) with a prescribed fixed point set. If $A \neq \emptyset$ or $A = X$, then every closed and nonempty subset F of X can be realized as the fixed set of a deformation $f: X \to X$ (see [4, Theorem 3.1] and its proof), but we shall see that this is no longer true if $\emptyset \neq A \neq X$.

The realization of a closed set F as the fixed point set of a proximity map $f: (|K|, |L|) \to (|K|, |L|)$ of a pair of compact polyhedra can be obtained with the help of the track α of a homotopy between the identity and f if α has the properties (2.1). If $\bar{d} \leq 1$ is a bounded metric for $|K|$ and F a closed non-empty subset of $|K|$, then the deformation $g: (|K|, |L|) \to (|K|, |L|)$ given by

$$g(x) = \alpha(x, f(x), \bar{d}(x, F))$$

has the fixed point set $\operatorname{Fix} g = F \cup \operatorname{Fix} f$, and therefore g has F as its fixed point set if f is constructed so that $\operatorname{Fix} f \subset F$. (See [5] for a description of this technique in the case $|L| = \emptyset$.)

The next theorem states the conditions on $F \subset X$ such that F can be realized as the fixed point set of a deformation of a 2-dimensionally connected pair of compact polyhedra (X, A).

Theorem 5.1. *Let $(X, A) = (|K|, |L|)$ be a 2-dimensionally connected pair of compact polyhedra, let A_j be the components of A and let F be a closed subset of X. Then there exists a deformation $g: (X, A) \to (X, A)$ with F as its fixed point set if and only if F has the following properties*:

(i) $F \cap A_j \neq \emptyset$ *if* $\chi(A_j) \neq 0$,

(ii) $F \cap (X - \operatorname{Int} A) \neq \emptyset$ *if* $\chi(X) \neq \chi(A)$.

Proof. To show that the conditions (i) and (ii) are sufficient, we only have to prove that they imply that there exists a proximity map $f: (X, A) \to (X, A)$ with $\operatorname{Fix} f \subset F$.

If $\chi(A_j) = 0$ for every component A_j of A, then it follows from Theorem 4.1 (i), (ii) that f can be chosen so that $\operatorname{Fix} f = \emptyset$ if $\chi(X) = \chi(A)$ or $\operatorname{Fix} f = \{x_0\} \in F \cap (X - \operatorname{Int} A)$ if $\chi(X) \neq \chi(A)$. We can further use Theorem 4.1 (iii) to find f if $\chi(A_j) \neq 0$ for $j = 1, 2, \ldots, k$ (where $k \geq 1$) and $\chi(X) = \chi(A)$, for we can then select points $a_j \in F \cap A_j$ and prescribe $\operatorname{Fix} f = \{a_1, a_2, \ldots, a_k\}$. If, finally, $\chi(A_j) \neq 0$ for $j = 1, 2, \ldots, k$ (where $k \geq 1$) and $\chi(X) \neq \chi(A)$, then we distinguish two cases. In the first, where $F \cap \operatorname{Bd} A = \emptyset$, we choose $x_0 \in F \cap (X - A)$ as well as $a_j \in F \cap A_j$ and use Lemma 3.1 to obtain a proximity map $f: (X, A) \to (X, A)$ with $\operatorname{Fix} f = \{a_1, a_2, \ldots, a_k\}$ or $\operatorname{Fix} f = \{a_1, a_2, \ldots, a_k, x_0\}$. In the second, where $F \cap \operatorname{Bd} A \neq \emptyset$, we choose $a_j \in A_j$ and $x_1 \in F \cap \operatorname{Bd} A$ (the point x_1 can coincide with one of the points a_j) and use Theorem 2.2 and Lemma 3.2 to obtain a proximity map $f: (X, A) \to (X, A)$ with $\operatorname{Fix} f = \{a_1, a_2, \ldots, a_k, x_1\}$. Thus we always obtain a proximity map $f: (X, A) \to (X, A)$ with $\operatorname{Fix} f \subset F$, and hence the map g of the Theorem.

It remains to show that the conditions (i) and (ii) are necessary. As any deformation of (X, A) maps A_j onto itself, the necessity of (i) is clear. Now let us assume that $f: (X, A) \to (X, A)$ is a deformation with fixed point set $F \subset \operatorname{Int} A$. Then it follows from the axioms of the fixed point index (see [1, p. 52] or [3, p. 11 ff]) that

$$\operatorname{ind}(f, \operatorname{Int} A_j) = \operatorname{ind}(\bar{f}, \operatorname{Int} A_j) \quad \text{for all } j = 1, 2, \ldots, k,$$

and hence

$$\chi(X) = \operatorname{ind}(f, X) = \sum_{j=1}^{k} \operatorname{ind}(f, \operatorname{Int} A_j)$$

and

$$\chi(A) = \sum_{j=1}^{k} \chi(A_j) = \sum_{j=1}^{k} \operatorname{ind}(\bar{f}, \operatorname{Int} A_j)$$

imply $\chi(X) = \chi(A)$. Thus (ii) is a necessary condition. □

Remark 5.2. Even if arbitrary maps of pairs and not only deformations are used, it is in general not possible to omit the conditions (i) and (ii) of Theorem 5.1 if one wants to realize a closed subset of X as the fixed point set of a map of pairs, for it is possible to construct a 2-dimensionally connected pair of compact polyhedra (X, A) and a closed subset $F \subset X$ which satisfies only one of the conditions (i) and (ii) and cannot be the fixed point set of any map $g: (X, A) \to (X, A)$. The fact that condition (i) cannot be omitted follows from the case where A is connected and has the fixed point property, and the fact that (ii) cannot be omitted follows from the next example, in which we construct a pair (X, A) and a set F which satisfy all assumptions of Theorem 5.1 so that (i) holds but not (ii) and so that F cannot be the fixed point set of any map $g: (X, A) \to (X, A)$.

Example 5.3. Let $X = B^2$ be the disk $\{x \mid |x| \leq 1\}$ in the plane, let A be the annulus $\{x \mid \frac{1}{2} \leq |x| \leq 1\}$ and let $F = S^1$ be the circle $|x| = 1$ which bounds B^2. Then Theorem 5.1 (i) is vacuously satisfied as $\chi(A) = 0$, but $F \cap (B^2 - \operatorname{Int} A) = \emptyset$. Now let us assume, by way of contradiction, that $f: (X, A) \to (X, A)$ is a map with fixed point set F. Then it follows from the axioms of the fixed point index (see [1] or [3]) that

$$\operatorname{ind}(\bar{f}, \operatorname{Int} A) = \operatorname{ind}(\bar{f}, A) = L(\bar{f}) = 0,$$

and hence that the fixed point index of the map $f: X \to X$ is

$$\operatorname{ind}(f, X) = \operatorname{ind}(f, \operatorname{Int} A) = \operatorname{ind}(\bar{f}, \operatorname{Int} A) = 0.$$

But this contradicts the fact that every selfmap of $X = B^2$ has fixed point index 1.

6. Two examples

We illustrate the results of Theorem 4.1 and 5.1 with two examples. The first one shows how the different cases of these theorems can occur, the second one gives an application to even-dimensional manifolds.

Example 6.1. Let X be the unit ball $\{x \mid |x| \leq 1\}$ in Euclidean n-space $(n \geq 2)$ with m holes removed in such a way that all holes lie in $\{x \mid |x| < \frac{3}{4}\}$, and let A consist of the boundary of X together with the outer ring $\{x \mid \frac{3}{4} \leq |x| \leq 1\}$. Hence A has $m+1$ components which we label so that A_j, for $j = 1, 2, \ldots, m$, are the $(n-1)$-spheres bounding the holes and A_{m+1} is the outer ring. (See Fig. 3 for the case $m = n = 2$.) Then (X, A) is a 2-dimensionally connected pair of compact polyhedra and we have

$$\chi(X) = \begin{cases} 1-m & \text{if } n \text{ is even,} \\ 1+m & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\chi(A) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1+m & \text{if } n \text{ is odd.} \end{cases}$$

Therefore three different cases occur. In the first, where n is even and $m = 1$, the pair (X, A) admits a fixed point free deformation, and any closed subset of X can be realized as the fixed point set of a deformation of (X, A). In the second, where n is even and $m \neq 1$, a deformation of (X, A) with a minimal fixed point set has one fixed point which can lie anywhere in $X - \operatorname{Int} A = \operatorname{Cl}(X - A_{m+1})$. A closed subset F of X can be realized as the fixed point set of a deformation of (X, A) if and only if $F \cap \operatorname{Cl}(X - A_{m+1}) \neq 0$. In the final case, where $n \geq 3$ is odd, a deformation of (X, A) with a minimal fixed point set has $1+m$ fixed points, and each component of A must contain exactly one of them. A closed subset of X can be the fixed point set of a deformation of (X, A) if and only if it intersects each component of A.

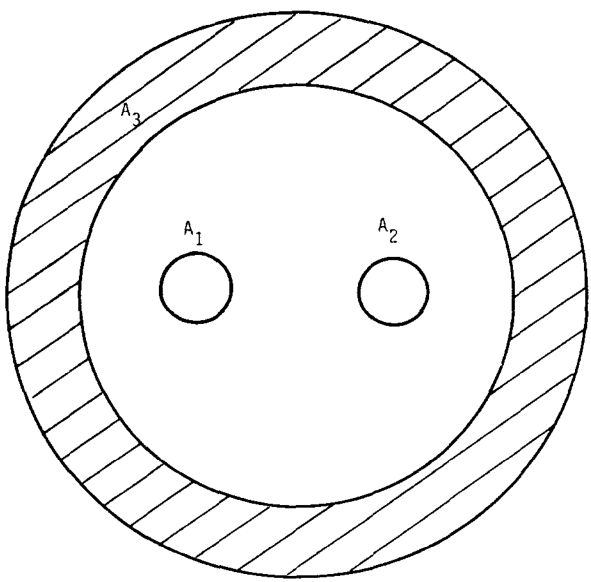

Fig. 3

Example 6.2. Let $X = M$ be a compact connected triangulable manifold of even dimension, and let $A = \mathrm{Bd}\, M$ be its boundary. Then each component A_j of A is a compact connected triangulable manifold without boundary of odd dimension, and hence $\chi(A_j) = 0$ and $\mathrm{Int}\, A = \emptyset$. Therefore Theorems 4.1 and 5.1 show that if $\chi(M) = 0$, then there exists a fixed point free deformation of $(M, \mathrm{Bd}\, M)$, and any closed subset of M can be the fixed point set of a deformation of $(M, \mathrm{Bd}\, M)$. If, on the other hand, $\chi(M) \neq 0$, then we see that there exists a deformation of $(M, \mathrm{Bd}\, M)$ with exactly one fixed point, and any closed and non-empty subset of M can be the fixed point set of a deformation of $(M, \mathrm{Bd}\, M)$. Hence the following theorem is a consequence of Theorems 4.1, 5.1 and [4, Theorem 3.1].

Theorem 6.3. *If M is a compact connected triangulable manifold of even dimension, then a closed subset of M can be the fixed point set of a deformation of $(M, \mathrm{Bd}\, M)$ if and only if it can be the fixed point set of a deformation of M.*

Note that Theorem 6.3 does not hold for odd-dimensional manifolds.

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